3.2 PROPERTIES OF DETERMINANTS.

We want to know if

a) $\det(A+B) \overset{?}{=} \det(A) + \det(B)$

b) $\det(AB) \overset{?}{=} \det(A)\det(B)$

c) $\det(kA) \overset{?}{=} k\det(A)$

for A, B $n \times n$ matrices and k scalar.

Before studying general results, let's consider some particular cases:

1) $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ 5 & -6 \end{bmatrix}$

$\Rightarrow \quad A+B = \begin{bmatrix} 3 & 0 \\ 7 & -2 \end{bmatrix}$

and $\det(A) = -2$, $\det(B) = 3$, $\det(A+B) = -6$

$\boxed{\det(A+B) \neq \det(A) + \det(B)}$

Therefore $\det(A+B) = -6 \neq \det(A) + \det(B) = -2 + 3 = 1$

---

2) However, for $A_2 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, $B_2 = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix}$, and

$C = \begin{bmatrix} 1 & 3 \\ 1 & 9 \end{bmatrix}$, we have

$\det(A_2) = -2$, $\det(B_2) = 8$, $\det(C) = 6$

Thus $\boxed{\det(C) = \det(A_2) + \det(B_2).}$

3) Also,

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 5 & -6 \end{bmatrix} = \begin{bmatrix} 17 & -21 \\ 24 & -30 \end{bmatrix}$$

$\det(A) = -2$, $\det(B) = 3$, $\det(AB) = -510 + 504 = -6$

Therefore,

$\boxed{\det(AB) = \det(A) \det(B)}$

$\det(AB) = -6 = (-2) * (3) = \det(A) \det(B)$.

---

4) Consider

$$3A = \begin{bmatrix} 3 & 9 \\ 6 & 12 \end{bmatrix}$$

Then $\det(3A) = 36 - 54 = -18 \neq 3 * (-2) =$
$= 3 \det(A)$

So $\boxed{\det(3A) \neq 3 \det(A)}$

In fact,

$\det(3A) = -18 = 9 * (-2) = 3^2 \det(A)$.

---

5) If $D = \begin{bmatrix} 5*1 & 5*3 \\ 2 & 4 \end{bmatrix}$

$\Rightarrow \det(D) = 5(1*4) - 5(3*2) = 5 * (4-6) =$
$= 5 * (-2) = 5 \det(A)$.

Does it mean $\boxed{\det \begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = K \det(A)}$?

In this section, we also want to generalize thm of section 2.2, it means

$$\boxed{A_{2\times 2} \text{ is invertible if and only if } \det A \neq 0,}$$

to general $n\times n$ matrices.

In order to do this, we need to learn how to compute $\det A$ from the determinants of the elementary matrices used in the row reduction of matrix A to an echelon form.

Relationship between det A and det B when B is obtained by performing an elementary row operation on A.

Consider the following cases: for $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

A) Multiplication of a row by scalar K.

$$B = \begin{bmatrix} \boxed{(5)(1) \quad (5)(3)} \\ 2 \quad\quad 4 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ 2 & 4 \end{bmatrix}$$

$\det B = 20 - 30 = -10 =$

$\boxed{\det A = 4 - 6 = -2}$

then, $\det B = -10 = 5(-2) = 5 \det A$

or $\boxed{\det B = 5 \det A}$

B) Interchanging two rows:

If $B = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \Rightarrow \det B = 6 - 4 = 2$

then, $\boxed{\det B = -(-2) = -\det A}$

c) <u>Replacement</u>
Adding a multiple of one row to another row:

$$B = \begin{bmatrix} 1+3(2) & 3+3(4) \\ 2 & 4 \end{bmatrix} \quad R_1' = R_1 + 3R_2$$

$$\Rightarrow \quad B = \begin{bmatrix} 7 & 15 \\ 2 & 4 \end{bmatrix}$$

$$\det B = 28 - 30 = -2$$

then, $\boxed{\det B = -2 = \det A}$

Relationship between $\det A$ and $\det B$, where B is obtained from A by row reduction.

Exercise

$$A = \begin{bmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{bmatrix} \xrightarrow{\text{3 replacements}} \sim \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 12 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow{\text{1 replac.}} \sim \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 108 & 23 \end{bmatrix} \xrightarrow{\text{1 replac.}} \sim \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & -13 \end{bmatrix} = \underline{\underline{B}}$$

How are $\det(A)$ and $\det(B)$ related?

$$\det(B) = 1 \cdot 1 \cdot (-3) \cdot (-13) = 39 \quad \text{thm 2}$$

$$\det(A) = \det(B) \quad \text{thm 3 (Next page)}$$

Observation: In this particular example, the only row operations are "replacements". If row interchanges or scaling were performed then it would not be longer true that

$\det A = \det B$ in general.

Thm 3 A is an $n \times n$ matrix

a) If B is the matrix that results when a row or a column is multiplied by k.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} ka_{11} & \cdots & ka_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

then, $\boxed{\det(B) = k\det(A).}$

b) If B is obtained from A by interchanging two rows or two columns, then $\boxed{\det(B) = -\det(A)}$

c) If B results by adding a multiple of one row to another row of A (Similar for Columns), then

$$\boxed{\det(B) = \det(A).}$$

(look at book's table for 3x3 matrices).

5

Anton : Linear Algebra

EXAMPLE 2 Theorem 3 Applied to 3 × 3 Determinants

Relationship	Operation
$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = k\det(A)$	The first row of A is multiplied by k.
$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = -\det(A)$	The first and second rows of A are interchanged.
$\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = \det(A)$	A multiple of the second row of A is added to the first row.

We will verify the equation in the last row of the table and leave the first two for the reader. With the help of Example 7 in Section 2.1 we obtain

$$\begin{aligned}
\det(B) &= (a_{11}+ka_{21})a_{22}a_{33} + (a_{12}+ka_{22})a_{23}a_{31} + (a_{13}+ka_{23})a_{21}a_{32} \\
&\qquad - a_{31}a_{22}(a_{13}+ka_{23}) - a_{33}a_{21}(a_{12}+ka_{22}) - a_{32}a_{23}(a_{11}+ka_{21}) \\
&= \det(A) + k(a_{21}a_{22}a_{33} + a_{22}a_{23}a_{31} + a_{23}a_{21}a_{32} \\
&\qquad - a_{31}a_{22}a_{23} - a_{33}a_{21}a_{22} - a_{32}a_{23}a_{21}) \\
&= \det(A) + 0 = \det(A) \qquad \blacklozenge
\end{aligned}$$

REMARK. As illustrated by the first equation in Example 2, part (*a*) of Theorem 2.2.3 allows us to bring a "common factor" from any row (or column) through the determinant sign.

Elementary Matrices Recall that an elementary matrix results from performing a single elementary row operation on an identity matrix; thus, if we let $A = I_n$ in Theorem 2.2.3 [so that we have $\det(A) = \det(I_n) = 1$], then the matrix B is an elementary matrix, and the theorem yields the following result about determinants of elementary matrices.

Theorem

Let E be an $n \times n$ elementary matrix.

(*a*) *If E results from multiplying a row of I_n by k, then $\det(E) = k$.*
(*b*) *If E results from interchanging two rows of I_n, then $\det(E) = -1$.*
(*c*) *If E results from adding a multiple of one row of I_n to another, then $\det(E) = 1$.*

Proof. of thm 3 part (a)

Let
$$B = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$$

Applying Cofactor expansion along first row to Compute det B leads to

$$\begin{aligned} \det B &= ka_{11}C_{11} + ka_{12}C_{12} + \cdots + ka_{1n}C_{1n} \\ &= k\left(a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}\right) \\ &= k \det A. \end{aligned}$$

Proof of Theorem 3 part (b) for a 3×3 matrix A.

For a 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Prove the following: Row interchange

$$\det A = -\det \underbrace{\begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{=B} = -\det B$$

Proof:- Renaming the entries of B

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

where
$b_{11} = a_{21}$, $b_{12} = a_{22}$, $b_{13} = a_{23}$
$b_{21} = a_{11}$, $b_{22} = a_{12}$, $b_{23} = a_{13}$
$b_{31} = a_{31}$, $b_{32} = a_{32}$, $b_{33} = a_{33}$

then

$$\det B = b_{11} b_{22} b_{33} + b_{21} b_{32} b_{13} + b_{12} b_{23} b_{31} - \left(b_{13} b_{22} b_{31} + b_{12} b_{21} b_{33} + b_{23} b_{32} b_{11} \right)$$

$$= a_{21} a_{12} a_{33} + a_{11} a_{32} a_{23} + a_{22} a_{13} a_{31} - \left(a_{23} a_{12} a_{31} + a_{22} a_{11} a_{33} + a_{13} a_{32} a_{21} \right) =$$

$$= -\left(a_{11} a_{22} a_{33} + a_{21} a_{32} a_{13} + a_{12} a_{23} a_{31} - \left(a_{13} a_{22} a_{31} + a_{12} a_{21} a_{33} + a_{23} a_{32} a_{11} \right) \right)$$

$$= -\det A.$$

<u>Lemma I:</u> If $B_{n\times n}$ matrix is such that two of its rows are identical then

$$\det B = 0.$$

<u>Proof:</u>- It is done for a 3×3 matrix. For the general case similar ideas can be applied.

Let

$$B = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then using cofactor expansion along first row

$$\det B = a_{21} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{22} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{23} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= \cancel{a_{21} a_{22} a_{33}} - \cancel{a_{21} a_{23} a_{32}} - \cancel{a_{22} a_{21} a_{33}} + \cancel{a_{22} a_{23} a_{31}} + \cancel{a_{23} a_{21} a_{32}} - \cancel{a_{23} a_{22} a_{31}}$$

$$= 0$$

Thus, $\boxed{\det B = 0}$

Lemma 2.- Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Then

$$\det \underbrace{\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}}_{=D} = \det A + \det C.$$

Proof.- Applying Cofactor expansion along the first row of D to compute its determinant leads to

$$\det D = (a_{11}+b_{11})C_{11} + (a_{12}+b_{12})C_{12} + \cdots + (a_{1n}+b_{1n})C_{1n}$$

$$= (a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}) + (b_{11}C_{11} + b_{12}C_{12} + \cdots + b_{1n}C_{1n})$$

$$= \det A + \det C$$

Proof of theorem 3 part (c).

Let

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & & \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} a_{11}+Ka_{21} & \cdots & a_{1n}+Ka_{2n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

then, using Lemma 2

$$\det B = \det A + \det \begin{bmatrix} Ka_{21} & \cdots & Ka_{2n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & & a_{nn} \end{bmatrix}$$

$$\overset{\text{part (a) Thm 3}}{=\!=} \det A + K \det \begin{bmatrix} a_{21} & \cdots & a_{2n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \overset{\text{lemma 1}}{=\!=} \det A + K \times 0$$

$$= \det A.$$

Therefore, $\det A$ does not change if a multiple of a row is added to another row.

Assume A has been reduced to an echelon form U by row replacements and r row interchanges : $\boxed{A \sim U}$

Then, from thm 3

$$\det A = (-1)^r \det U$$

Since U is in echelon form and $U_{n\times n}$ is a triangular matrix

$$\det U = u_{11} \cdots u_{nn} \quad (\text{Triangular matrix})$$

Then

$$\det A = \begin{cases} (-1)^r \left(\begin{matrix}\text{product of pivots}\\ \text{in } U\end{matrix}\right), & A \text{ invertible} \\ 0, & A \text{ is not invertible} \end{cases}$$

Jump to 12.

Now, we want to establish a relationship between invertibility of matrices and their determinants.

First, we will learn how to compute $\det A$ from determinants of elementary matrices.

Corollary 1 from thm 3.-

Let E be an $n \times n$ elementary matrix

a) If E results from multiplying a row of I_n by k, then $\det E = k$.

b) If E results from interchanging two rows of I_n, then $\det E = -1$.

c) If E results from adding a multiple of one row of I_n to another row, then

$$\det E = 1.$$

Proof.- It is an immediate consequence of thm 3 and the fact that

$$\det I_n = 1.$$

Corollary 2 from thm 3

If $B_{n\times n}$ matrix and $E_{n\times n}$ elementary matrix, then

$$\boxed{\det EB = \det E \det B}$$

Proof.- There are three cases depending on E.

Case 1.- $E = \begin{bmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & k & & \\ & 0 & & \ddots & \\ & & & & 1 \end{bmatrix}$ i^{th} row , $\det E \overset{\text{Corollary 1 (a)}}{=} k$

then $EB = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & & & \\ kb_{i1} & kb_{i2} & \cdots & kb_{in} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$

and $\det EB \overset{\text{thm 3}}{=} k \det B = \det E \det B$

Case 2.-

If E is obtained from interchanging two rows of I_n

then $\det E \overset{\text{Corollary 1 (b)}}{=} -1$ and

EB is the matrix that results from interchanging the same rows of B.

Therefore, $\det EB \overset{\text{Thm 3}}{=} -\det B = \det E \det B$.

Case 3.-

If E is obtained by adding a multiple of one row of I_n to another row i then

$$\det E \overset{\text{Corollary 1 (c)}}{=} 1$$

and EB is the matrix that results from adding a multiple of the same row of B to the row i of B.

Therefore,

$$\det EB \overset{\text{thm 3}}{=\!=} \det B = \det E \det B.$$

Corollary 3.- If $B_{n\times n}$ matrix and $E_1, E_2, .., E_r$ are $n\times n$ elementary matrices,

then

$$\boxed{\det(E_1 E_2 E_3 .. E_r B) = \det E_1 \det E_2 \dots \det E_r \det B.}$$

Proof.-

$$\det(E_1 E_2 . E_r B) \overset{\text{Cor. 2}}{=} \det E_1 \det(E_2 E_3 \dots E_r B)$$

$$\det E_1 \det E_2 \det(E_3 \dots E_r B) = \dots \overset{\text{Cor 2.}}{=} \det E_1 \det E_2 .. \det E_r \det B$$

Thm 4 $A_{n\times n}$ is invertible if and only if $\det A \neq 0$.

Proof.-

Jump to (13)

$(\rightarrow)$ A invertible implies that

$$A = E_1 E_2 \dots E_r I_n \quad (\text{Inv. matrix thm}).$$

Then using Corollary 3

$$\det A = \overset{\neq 0}{\det E_1}\, \overset{\neq 0}{\det E_2} \dots \overset{\neq 0}{\det E_r}\, \overset{=1}{\det I_n} \neq 0$$

$(\leftarrow)$ Consider R the row-reduced echelon form of A then $R = I_n$ or R has a row of zeros (Key lemma)

Also $\quad R = E_1 E_2 \dots E_r A$

Therefore, $\det R \overset{\text{Cor. } 3}{=} \overset{\neq 0}{\det E_1}\, \overset{\neq 0}{\det E_2} \dots \overset{\neq 0}{\det E_r}\, \overset{\neq 0}{\det A} \neq 0$

Then, R cannot have a row of zeros, otherwise

$$\det R = 0.$$

Therefore, the only possibility for R is to be I_n

$$\text{i.e.,} \quad R = I_n.$$

and that means $A \sim R = I$ is invertible

according to Inv. matrix thm.

Previous to thm 6

$$\det(EA) \overset{\text{thm } 3}{=} \det(E)\det(A).$$

Thm 6.– $A_{n\times n}$ and $B_{n\times n}$

$$\det AB = \det A \det B$$

Proof.– Two cases arise depending on A.

CASE 1.– A is singular $\Longrightarrow$ AB is singular. (Statement in pag 6 Sect. 2.3)

$$\Rightarrow \det AB = 0 = 0 . \det B = \det A \det B$$

CASE 2.– A is invertible $\overset{IMT}{\Longrightarrow} A \sim I_n$

$$\Rightarrow E_r \ldots E_1 A = I_n \xRightarrow{\substack{\text{Using inverse of}\\ \text{elem. matrices}}} A = E_1^{-1} \ldots E_r^{-1} \quad \text{also elem. matrices.}$$

$$\Rightarrow \det AB = \det\left(E_1^{-1} \ldots E_r^{-1} B\right) \overset{\substack{\text{Corollary}\\ 2}}{=} \det E_1^{-1} \det\left(E_2^{-1} \ldots E_r^{-1} B\right) =$$

$$\overset{\substack{\text{after } r-1\\ \text{steps}}}{=} \det E_1^{-1} \det E_2^{-1} \ldots \det E_r^{-1} \det B =$$

$$\overset{\text{Corollary 3}}{=} \det\left(E_1^{-1} E_2^{-1} \ldots E_r^{-1}\right) \det B . = \det A \det B$$